Monodromy Groups of Belyĭ Lattès Maps

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Abstract

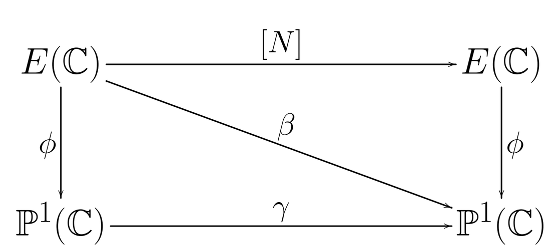
A rational map $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ from the Riemann Sphere to itself is said to be a Lattès Map if there are "well-behaved" maps $\phi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and $\psi : E(\mathbb{C}) \to E(\mathbb{C})$ such that $\gamma \circ \phi = \phi \circ \psi$. We are interested in those Lattès Maps which are also Belyĭ Maps and their associated monodromy groups. This work is conducted as part of the Pomona Research in Mathematics Experience (DMS-2113782).

Elliptic Curves

- An elliptic curve is an equation of the form $y^2 = x^3 + A\,x + B$ where $4\,A^3 + 27\,B^2 \neq 0$. Equivalently, it is a non-singular curve of genus one.
- Let $S = E(\mathbb{C})$ be the collection of complex numbers x_0 and y_0 satisfying $y^2 = x^3 + A\,x + B$ along with the "point at infinity" O_E. This is a torus. In particular, it is a compact, connected Riemann surface.
- Let P, Q, and $P * Q$ be points on E which lie on a line. Then the binary operation $P \oplus Q = (P * Q) * O_E$ turns $(E(\mathbb{C}), \oplus)$ into an abelian group.

Belyĭ Lattès Maps

- A Belyĭ map $\phi : S \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic function defined on a compact, connected Riemann surface S which is ramified over at most three points. We choose these points to be 0, 1, and ∞.
- A Lattès map $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic function satisfying $\gamma \circ \phi = \phi \circ [N]$ for some meromorphic function $\phi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and "multiplication-by-N" $[N] : E(\mathbb{C}) \to E(\mathbb{C})$ where $[N]P = P \oplus \cdots \oplus P$.



Theorem (Ayberk Zeytin, 2021; PRiME 2022)

Assume $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Lattès map.

- If ϕ is a Belyĭ map, then both γ and $\beta = \gamma \circ \phi$ are Belyĭ maps as well.
- Any Belyĭ Lattès map arises from one of three families:

Elliptic Curve	Belyĭ Map ϕ	Degree of ϕ
$E : y^2 = x^3 + B$	$\phi(x,y) = \dfrac{\sqrt{B} - y}{2\sqrt{B}}$	$\deg(\phi) = 3$
$E : y^2 = x^3 + A\,x$	$\phi(x,y) = -\dfrac{x^2}{A}$	$\deg(\phi) = 4$
$E : y^2 = x^3 + B$	$\phi(x,y) = -\dfrac{x^3}{B}$	$\deg(\phi) = 6$

Dessin d'Enfants

Assume that $\phi : S \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ map for a compact, connected Riemann surface S whose critical values are 0, 1, and ∞. A Dessin d'Enfants is a bipartite graph whose "black" vertices are the preimage $\phi^{-1}(\{0\})$, "red" vertices are the preimage $\phi^{-1}(\{1\})$, and "edges" are the preimage $\phi^{-1}((0,1))$.

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Monodromy Groups

Assume that $\phi : S \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ map of degree M. The monodromy group of ϕ is a subgroup of the symmetric group of degree M as follows:

1. Label the edges of the Dessin d'Enfants of ϕ using 1 through M.
2. Write down the permutation σ_0 as the product of disjoint cycles found from reading the labels counterclockwise around each "black" vertex.
3. Write down the permutation σ_1 as the product of disjoint cycles found from reading the labels counterclockwise around each "red" vertex.
4. Generate the group $\mathrm{Mon}(\gamma) = \langle \sigma_0, \sigma_1 \rangle$ from these permutations.

Examples

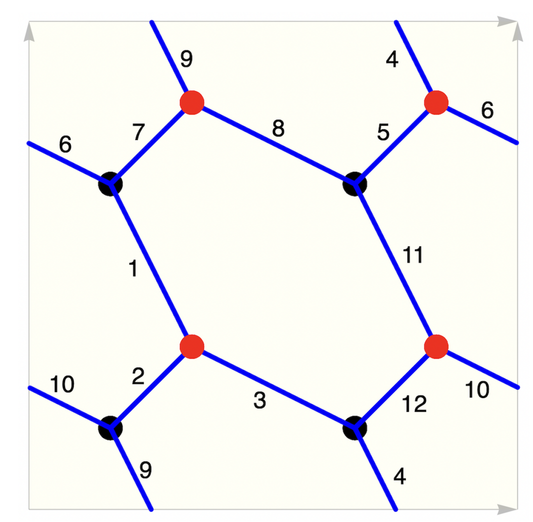
Consider $E : y^2 = x^3 + 1$ and $\phi(x,y) = (1-y)/2$.

- Denote the Belyĭ map

$$\beta(x,y) = \phi \circ [2] = \frac{(1+y)\,(3-y)^3}{16\,y^3}.$$

We compute its monodromy as $\mathrm{Mon}(\beta) = \langle \sigma_0, \sigma_1 \rangle \simeq A_4$.

$$\sigma_0 = (1\;7\;6)\,(2\;10\;9)\,(3\;4\;12)\,(5\;8\;11)$$
$$\sigma_1 = (1\;2\;3)\,(4\;5\;6)\,(7\;8\;9)\,(10\;11\;12)$$

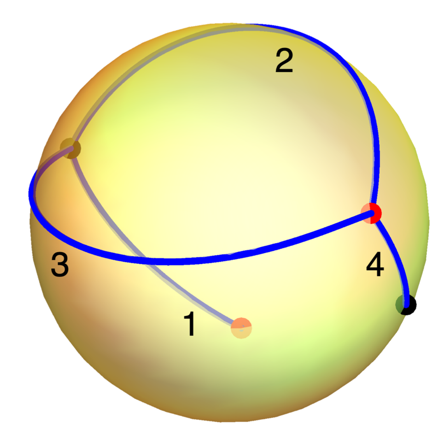


- Denote the Belyĭ Lattès map

$$\gamma = \frac{(z-1)\,(z+1)^3}{(2\,z-1)^3},$$

so that $\beta = \gamma \circ \phi = \phi \circ [2]$. We compute $\mathrm{Mon}(\gamma) \simeq A_4$ as well.

$$\sigma_0 = (4)\,(1\;3\;2)$$
$$\sigma_1 = (1)\,(2\;3\;4)$$



Motivating Questions

- These examples of monodromy groups suggest $\mathrm{Mon}(\beta)$ and $\mathrm{Mon}(\gamma)$ are equal in certain cases. Under which conditions is this true?
- More generally, what is the precise relationship between $\mathrm{Mon}(\beta)$ and $\mathrm{Mon}(\gamma)$?

Theorem (PRiME 2022)

Assume $\phi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is Belyĭ map of degree M for an elliptic curve $E : y^2 = x^3 + A\,x + B$ satisfying the following two assumptions:

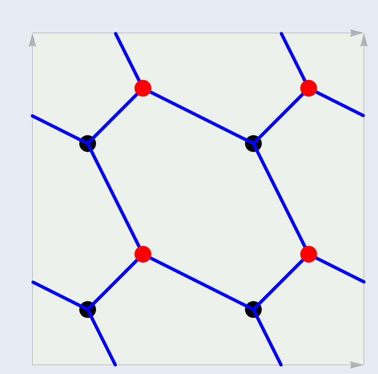
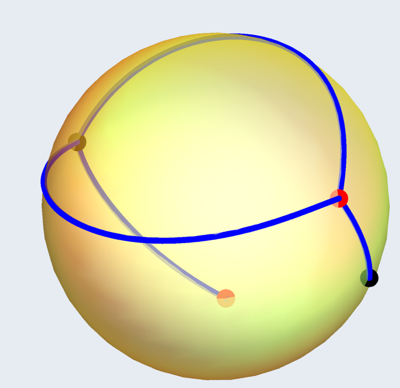
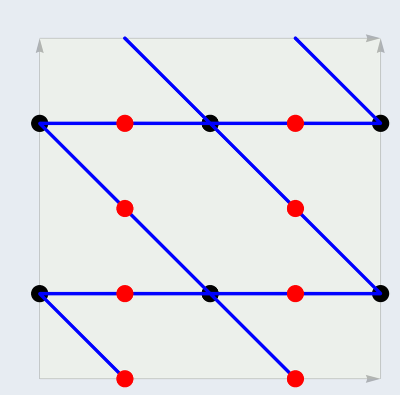
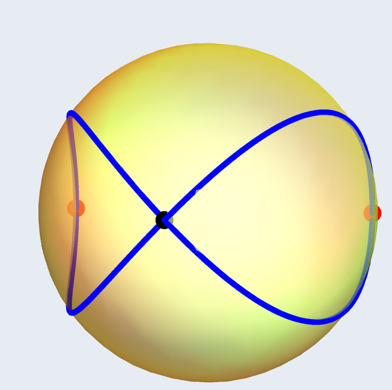
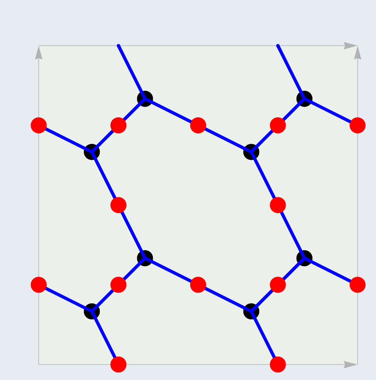
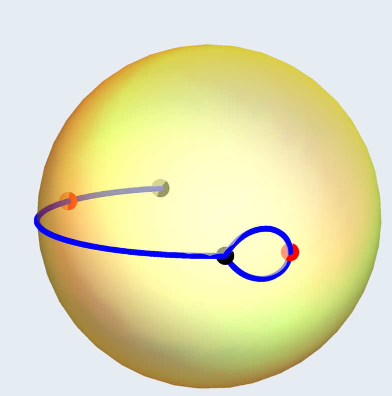
i. The group homomorphism $\mathbb{Z}/M\mathbb{Z} \to \mathrm{Mon}(\phi)$ sending $m \bmod M$ to $[\zeta^m](x,y) = (\zeta^{2m}\,x,\ \zeta^{3m}\,y)$ is an isomorphism for $\zeta = e^{2\pi i/M}$.

ii. For each $N = 1, 2, 3, \ldots$, there exists some meromorphic function $\gamma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\gamma \circ \phi = \phi \circ [N]$.

Then we have the following:

1. The three families in Zeytin's classification satisfy these assumptions.
2. The composition $\beta = \gamma \circ \phi = \phi \circ [N]$ is a Belyĭ map of degree N^2M. Explicitly, $P \mapsto [\zeta^m]P \oplus P_0$ is an automorphism for $P_0 \in E[N]$ and $m \in \mathbb{Z}/M\mathbb{Z}$. In particular, we have the monodromy group
$$\mathrm{Mon}(\beta) \simeq E[N] \rtimes \mathrm{Mon}(\phi) \simeq \left(\frac{\mathbb{Z}}{N\mathbb{Z}} \times \frac{\mathbb{Z}}{N\mathbb{Z}}\right) \rtimes \left(\frac{\mathbb{Z}}{M\mathbb{Z}}\right).$$
3. γ is a Belyĭ Lattès map of degree N^2 with monodromy group
$$\mathrm{Mon}(\gamma) \simeq \left(\frac{\mathbb{Z}}{N\mathbb{Z}} \times \frac{\mathbb{Z}}{N\mathbb{Z}}\right) \rtimes \left(\frac{d\,\mathbb{Z}}{M\mathbb{Z}}\right)$$
where
$$d = [\mathrm{Mon}(\beta) : \mathrm{Mon}(\gamma)] = \begin{cases} M & \text{if } N = 1, \\ 2 & \text{if } N = 2 \text{ and } M \text{ is even, and} \\ 1 & \text{otherwise.} \end{cases}$$

Examples of Dessins d'Enfants of Belyĭ Maps and Belyĭ Lattès Maps

$\deg(\phi)$	Belyĭ Map $\beta = \phi \circ [2]$	Dessin d'Enfants of β	Belyĭ Lattès Map γ	Dessin d'Enfants of γ
3	$\beta(x,y) = \dfrac{(1+y)\,(3-y)^3}{16\,y^3}$		$\gamma(z) = \dfrac{(z-1)\,(z+1)^3}{(2\,z-1)^3}$	
4	$\beta(x,y) = \dfrac{(x^2+1)^4}{16\,x^2\,(x^2-1)^2}$		$\gamma(z) = \dfrac{(z+1)^4}{16\,z\,(z-1)^2}$	
6	$\beta(x,y) = -\dfrac{x^3\,(x^3-8)^3}{64\,(x^3+1)^3}$		$\gamma(z) = \dfrac{z\,(z+8)^3}{64\,(z-1)^3}$	

How Do We Generate Belyĭ Lattès Maps?

$\deg(\phi)$	Substitutions	Belyĭ Lattès Map $\gamma(z)$
3	$x = \left(4\,z\,(z-1)\right)^{1/3}$ $y = 1 - 2\,z$	$\dfrac{1}{2\sqrt{B}}\left(\sqrt{B} - \dfrac{\psi_{2N}(x,y)}{2\,\psi_N(x,y)^4}\right)$
4	$x = z^{1/2}$ $y = z^{1/4}\,(z-1)^{1/2}$	$-\dfrac{1}{A}\left(x - \dfrac{\psi_{N+1}(x,y)\,\psi_{N-1}(x,y)}{\psi_N(x,y)^2}\right)^2$
6	$x = -z^{1/3}$ $y = (1-z)^{1/2}$	$-\dfrac{1}{B}\left(x - \dfrac{\psi_{N+1}(x,y)\,\psi_{N-1}(x,y)}{\psi_N(x,y)^2}\right)^3$

Sketch of Proof

We proved these results using Galois theory by relating $\mathrm{Mon}(\gamma)$ to the largest normal subgroup of $G \simeq \mathrm{Mon}(\beta)$ which is contained in $H \simeq \mathrm{Mon}(\phi)$.

$$L = \mathcal{K}(E(\mathbb{C})) \qquad\qquad \{1\}$$
$$M \qquad\qquad M$$
$$K = \mathbb{C}(\phi(x,y)) \qquad\qquad H = \mathrm{Gal}(L/K) \simeq \frac{\mathbb{Z}}{M\mathbb{Z}} \simeq \mathrm{Mon}(\phi)$$
$$N^2 \qquad\qquad N^2$$
$$F = \mathbb{C}(\beta(x,y)) \qquad\qquad G = \mathrm{Gal}(L/F) \simeq E[N] \rtimes \mathrm{Mon}(\phi) \simeq \mathrm{Mon}(\beta)$$

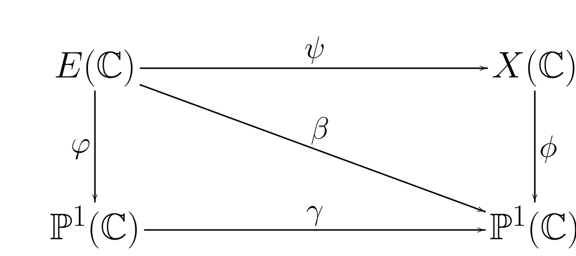
Explicitly, $\mathrm{Mon}(\gamma)$ is the quotient of $G \simeq E[N] \rtimes \mathrm{Mon}(\phi)$ by

$$\bigcap_{\sigma \in G} (\sigma H \sigma^{-1}) \simeq \left\{ n \in \frac{\mathbb{Z}}{M\mathbb{Z}} \;\middle|\; [\zeta^n]P_0 = P_0 \text{ for all } P_0 \in E[N] \right\} \simeq \frac{\mathbb{Z}}{d\,\mathbb{Z}}.$$

Future Work

We now understand the relationship between the monodromy groups of the composition $\beta = \gamma \circ \phi = \phi \circ [N]$ and the Belyĭ Lattès map γ for a Belyĭ map $\phi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and the "multiplication-by-N" map $[N] : E(\mathbb{C}) \to E(\mathbb{C})$.

We wish to generalize and understand the relationship between the monodromy groups of $\beta = \gamma \circ \varphi = \varphi \circ \psi$ and γ for Belyĭ maps $\varphi : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and $\phi : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ and an arbitrary isogeny $\psi : E(\mathbb{C}) \to X(\mathbb{C})$ between two elliptic curves E and X.



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